DYNAMIC SCREENING WITH LIMITED COMMITMENT
SUPPLEMENTARY MATERIAL

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This document contains the omitted analysis and proofs from Section 3 of Deb and Said (2015), "Dynamic Screening with Limited Commitment," *Journal of Economic Theory*, `http://dx.doi.org/10.1016/j.jet.2015.05.015`.

Appendix B. Omitted Analysis and Proofs from Section 3

The period-two incentive compatibility constraints (IC_{12}) and (IC_{22}) are "easy": each is essentially a static incentive compatibility constraint, and can therefore be reduced to the usual envelope and monotonicity conditions. The proof follows standard techniques and is therefore omitted.

Lemma B.1. *The period-two incentive compatibility constraints* (IC_{12}) *for buyers contracting in period one are satisfied if, and only if, for all* $\lambda \in \Lambda$,

$$\frac{\partial}{\partial v} U_{12}(v, \lambda) = q_1(v, \lambda) \text{ almost everywhere, and } q_1(v, \lambda) \text{ is nondecreasing in } v. \tag{IC'_{12}}$$

The constraints (IC_{22}) *for buyers contracting in period two are satisfied if, and only if,*

$$U'_{22}(v) = q_2(v) \text{ almost everywhere, and } q_2(v) \text{ is nondecreasing in } v. \tag{IC'_{22}}$$

Since the underlying mechanisms are deterministic, Lemma B.1 implies that the allocation rules must be cutoff policies: there exists a function $k_1 : \Lambda \to \mathbf{V}$ and a constant $\alpha \in \mathbf{V}$ such that

$$q_1(v, \lambda) = \begin{cases} 0 & \text{if } v < k_1(\lambda), \\ 1 & \text{if } v \geq k_1(\lambda); \end{cases} \text{ and } q_2(v) = \begin{cases} 0 & \text{if } v < \alpha, \\ 1 & \text{if } v \geq \alpha. \end{cases}$$

Lemma B.1 also immediately yields the observation that $\frac{\partial}{\partial v}\widetilde{U}_{12}(v, \lambda) = q_2(v)$ for all $\lambda \in \Lambda$ whenever (IC_{22}) is satisfied, as $\widetilde{U}_{12}(v, \lambda)$ directly inherits the properties of $U_{22}(v)$. We can use this result to characterize the period-two continuation payoff $V_{12}(v, \lambda)$ of a cohort-one buyer when we impose constraint ($\widetilde{\text{RC}}$) and allow buyers recontract in period two.

Lemma B.2. *Define* $\bar{q}_{12}(v, \lambda) := x_2(v, \lambda)q_1(v, \lambda) + (1 - x_2(v, \lambda))q_2(v)$ *for all* $\lambda \in \Lambda$ *and* $v \in \mathbf{V}$, *and suppose* (IC_{12}), (IC_{22}), *and* ($\widetilde{\text{RC}}$) *are satisfied. Then for all* $\lambda \in \Lambda$,

$$\frac{\partial}{\partial v} V_{12}(v, \lambda) = \bar{q}_{12}(v, \lambda) \text{ almost everywhere, and } \bar{q}_{12}(v, \lambda) \text{ is nondecreasing in } v.$$

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In addition, for all $\lambda \in \Lambda$, $\bar{q}_{12}(v,\lambda) = q_2(v)$ *for almost all* $v \in \mathbf{V}$ *such that* $x_2(v,\lambda) \in (0,1)$. *Finally,* $\bar{q}_{12}(v,\lambda)$ *corresponds to a cutoff policy with threshold* $\bar{k}_{12}(\lambda)$ *for all* $\lambda \in \Lambda$.

PROOF. Suppose that ($\widetilde{\mathrm{RC}}$) is satisfied. Then

$$V_{12}(v,\lambda) = \max_{v'} \left\{ x_2(v',\lambda)[q_1(v',\lambda)v - p_{12}(v',\lambda)] + (1-x_2(v',\lambda))[q_2(v')v - p_{22}(v') + \tilde{p}_{12}(\lambda)] \right\}.$$

Applying the Envelope Theorem—see Milgrom and Segal (2002)—yields

$$\frac{\partial}{\partial v} V_{12}(v,\lambda) = x_2(v,\lambda)q_1(v,\lambda) + (1-x_2(v,\lambda))q_2(v) = \bar{q}_{12}(v,\lambda).$$

Moreover, note that we can rewrite the optimality condition above as

$$V_{12}(v,\lambda) \geq V_{12}(v',\lambda) + \bar{q}_{12}(v',\lambda)(v-v') \text{ for all } v,v' \in \mathbf{V}.$$

A similar inequality is derived by interchanging the role of v and v'. Adding these two inequalities yields $(\bar{q}_{12}(v,\lambda) - \bar{q}_{12}(v',\lambda))(v-v') \geq 0$, so $\bar{q}_{12}(v,\lambda)$ nondecreasing in v for all $\lambda \in \Lambda$.

Now fix any $v \in \mathbf{V}$ such that $x_2(v,\lambda) \in (0,1)$ and v is a point of differentiability for $V_{12}(\cdot,\lambda)$. Recall first that Lemma B.1 implies that $q_1(\cdot,\lambda)$ and q_2 correspond to cutoff rules, with cutoffs $k_1(\lambda)$ and α respectively. So suppose that $k_1(\lambda) > \alpha$, and note that we can write

$$\bar{q}_{12}(v,\lambda) = \begin{cases} x_2(v,\lambda)(1) + (1-x_2(v,\lambda))(1) = 1 = q_2(v) & \text{if } v > k_1(\lambda); \\ x_2(v,\lambda)(0) + (1-x_2(v,\lambda))(1) = 1 - x_2(v,\lambda) < q_2(v) & \text{if } v \in (\alpha, k_1(\lambda)); \\ x_2(v,\lambda)(0) + (1-x_2(v,\lambda))(0) = 0 = q_2(v) & \text{if } v < \alpha. \end{cases}$$

Thus, if $v \in (\alpha, k_1(\lambda))$, we have $\frac{\partial}{\partial v} V_{12}(v,\lambda) < \frac{\partial}{\partial v} \widetilde{U}_{12}(v,\lambda)$, implying that ($\widetilde{\mathrm{RC}}$) is violated for some $v' \in (v, v+\epsilon)$ for $\epsilon > 0$ sufficiently small. Similarly, suppose that $k_1(\lambda) < \alpha$. Then

$$\bar{q}_{12}(v,\lambda) = \begin{cases} x_2(v,\lambda)(1) + (1-x_2(v,\lambda))(1) = 1 = q_2(v) & \text{if } v > \alpha; \\ x_2(v,\lambda)(1) + (1-x_2(v,\lambda))(0) = x_2(v,\lambda) > q_2(v) & \text{if } v \in (k_1(\lambda), \alpha); \\ x_2(v,\lambda)(0) + (1-x_2(v,\lambda))(0) = 0 = q_2(v) & \text{if } v < k_1(\lambda). \end{cases}$$

Thus, if $v \in (k_1(\lambda), \alpha)$, we have $\frac{\partial}{\partial v} V_{12}(v,\lambda) > \frac{\partial}{\partial v} \widetilde{U}_{12}(v,\lambda)$, implying that ($\widetilde{\mathrm{RC}}$) is violated for some $v' \in (v-\epsilon, v)$ for $\epsilon > 0$ sufficiently small. Thus, $\bar{q}_{12}(v,\lambda) \in \{q_1(v,\lambda), q_2(v)\}$ for almost all v.

Since both q_1 and q_2 are deterministic, this implies that $\bar{q}_{12}(v,\lambda) \in \{0,1\}$ almost everywhere. Of course, since $\bar{q}_{12}(v,\lambda)$ is nondecreasing in v for all λ, we can therefore treat it as a cutoff policy. ■

In order to simplify (IC_{11}), consider the "effective" allocation rule

$$\bar{q}_1(v,\lambda) := x_1(\lambda)x_2(v,\lambda)q_1(v,\lambda) + (1 - x_1(\lambda)x_2(v,\lambda))q_2(v) = x_1(\lambda)\bar{q}_{12}(v,\lambda) + (1-x_1(\lambda))q_2(v).$$

Clearly, $\bar{q}_1$ inherits monotonicity in v from $\bar{q}_{12}(v,\lambda)$ and $q_2(v)$. An envelope argument combines with the stochastic order on $\{G(\cdot|\lambda)\}_{\lambda\in\Lambda}$ to show that $\bar{q}_1$ must also be nondecreasing in λ.[1]

[1]The sufficiency of monotonicity for incentive compatibility follows from known results; see Pavan, Segal, and Toikka (2014) for the general formulation of this observation. Necessity, on the other hand, relies on the restriction (as in Krähmer and Strausz (2011)) to *deterministic* allocation rules.

LEMMA B.3. *Suppose that constraints* (SD), (IC_{12}), *and* (IC_{22}) *are satisfied. If the initial-period constraint* (IC_{11}) *is satisfied, then*

$$V'_{11}(\lambda) = -\int_{\mathbf{V}} \bar{q}_1(v,\lambda) G_\lambda(v|\lambda) dv \text{ almost everywhere, and} \tag{IC'_{11}}$$

$$\bar{q}_1(v,\lambda) \text{ is nondecreasing in } v \text{ and } \lambda. \tag{MON_{11}}$$

In addition, for almost all $\lambda \in \Lambda$ with $x_1(\lambda) \in (0,1)$, $\bar{q}_1(v,\lambda) = q_2(v)$ for all $v \in \mathbf{V}$. Finally, $\bar{q}_1(v,\lambda)$ corresponds to a cutoff policy with threshold $\bar{k}_1(\lambda)$ for all $\lambda \in \Lambda$.

PROOF. We begin by showing the necessity of (IC'_{11}). Note first that we can rewrite (IC_{11}) as

$$U_{11}(\lambda) = \max_{\lambda'} \left\{ -p_{11}(\lambda') + \int_{\mathbf{V}} V_{12}(v,\lambda') dG(v|\lambda) \right\}.$$

Applying the Envelope Theorem, integration by parts, and Lemma B.2 yields

$$U'_{11}(\lambda) = \int_{\mathbf{V}} V_{12}(v,\lambda) \frac{\partial g(v|\lambda)}{\partial \lambda} dv = -\int_{\mathbf{V}} \frac{\partial V_{12}(v,\lambda)}{\partial v} G_\lambda(v|\lambda) dv = -\int_{\mathbf{V}} \bar{q}_{12}(v,\lambda) G_\lambda(v|\lambda) dv.$$

In addition, note that we can use Lemma B.1's envelope formulation of (IC_{22}) to show that

$$\widetilde{U}'_{11}(\lambda) = \int_{\mathbf{V}} U_{22}(v) \frac{\partial g(v|\lambda)}{\partial \lambda} dv = -\int_{\mathbf{V}} \frac{\partial U_{22}(v)}{\partial v} G_\lambda(v|\lambda) dv = -\int_{\mathbf{V}} \bar{q}_2(v) G_\lambda(v|\lambda) dv.$$

Finally, notice that (SD) and (IC_{11}) jointly imply that

$$V_{11}(\lambda) = \max_{\lambda'} \left\{ x_1(\lambda') \left(-p_{11}(\lambda') + \int_{\mathbf{V}} V_{12}(v,\lambda') dG(v|\lambda) \right) + (1 - x_1(\lambda')) \int_{\mathbf{V}} U_{22}(v) dG(v|\lambda) \right\}.$$

Once again, the Envelope Theorem, along with the previous two observations, implies that

$$V'_{11}(\lambda) = x_1(\lambda) U'_{11}(\lambda) + (1 - x_1(\lambda)) \widetilde{U}_{11}(\lambda) = -\int_{\mathbf{V}} \bar{q}_1(v,\lambda) G_\lambda(v|\lambda) dv.$$

We now show that, for almost all $\lambda \in \Lambda$ such that $x_1(\lambda) \in (0,1)$, $\bar{q}_1(v,\lambda) = q_2(v)$. Fix any $\lambda \in (\underline{\lambda}, \bar{\lambda})$ such that $x_1(\lambda) \in (0,1)$ and V_{11} is differentiable at λ. Then we can write

$$V'_{11}(\lambda) = -\int_{\mathbf{V}} \bar{q}_1(v,\lambda) G_\lambda(v|\lambda) dv = -\int_{\mathbf{V}} q_2(v) G_\lambda(v|\lambda) dv - x_1(\lambda) \int_{\mathbf{V}} [\bar{q}_{12}(v,\lambda) - q_2(v)] G_\lambda(v|\lambda) dv.$$

Since $q_2(\cdot)$ and $\bar{q}_{12}(\cdot,\lambda)$ correspond to cutoffs α and $\bar{k}_{12}$, respectively, $V'_{11}(\lambda)$ can be rewritten as

$$V'_{11}(\lambda) = -\int_{\alpha}^{\bar{v}} G_\lambda(v|\lambda) dv - x_1(\lambda) \int_{\bar{k}_{12}(\lambda)}^{\alpha} G_\lambda(v|\lambda) dv = \widetilde{U}'_{11}(\lambda) - x_1(\lambda) \int_{\bar{k}_{12}(\lambda)}^{\alpha} G_\lambda(v|\lambda) dv.$$

Thus, if $\bar{k}_{12}(\lambda) < \alpha$, we have $V'_{11}(\lambda) > \widetilde{U}'_{11}(\lambda)$, implying (SD) is violated for some $\lambda' \in (\lambda - \epsilon, \lambda)$ for $\epsilon > 0$ sufficiently small. On the other hand, if $\bar{k}_{12}(\lambda) > \alpha$, then $V'_{11}(\lambda) < \widetilde{U}'_{11}(\lambda)$ and (SD) is violated for some $\lambda' \in (\lambda, \lambda + \epsilon)$ for sufficiently small $\epsilon > 0$. Thus, we must have $\bar{k}_{12}(\lambda) = \alpha$, or equivalently, $\bar{q}_{12}(v,\lambda) = \bar{q}_2(v)$ for all $v \in \mathbf{V}$. Therefore, we can conclude that

$$\bar{q}_1(v,\lambda) = x_1(\lambda) \bar{q}_{12}(v,\lambda) + (1 - x_1(\lambda)) q_2(v) = x_1(\lambda) q_2(v) + (1 - x_1(\lambda)) q_2(v) = q_2(v).$$

Finally, to see that (IC_{11}) and (SD) imply (MON_{11}), fix any $\lambda, \lambda' \in \Lambda$. We can write

$$V_{11}(\lambda) \geq x_1(\lambda')[-p_{11}(\lambda') + \int_{\mathbf{V}} V_{12}(v,\lambda') dG(v|\lambda)] + (1 - x_1(\lambda')) \widetilde{U}_{11}(\lambda)$$

$$
\begin{aligned}
&= x_1(\lambda')\left(U_{11}(\lambda') + \int_{\mathbf{V}} V_{12}(v,\lambda')d[G(v|\lambda) - G(v|\lambda')]\right)\\
&\quad + (1 - x_1(\lambda'))\left(\widetilde{U}_{11}(\lambda') + \int_{\mathbf{V}} U_{22}(v)d[G(v|\lambda) - G(v|\lambda')]\right)\\
&= V_{11}(\lambda') + \int_{\mathbf{V}}[x_1(\lambda')V_{12}(v,\lambda') + (1 - x_1(\lambda'))U_{22}(v)]d[G(v|\lambda) - G(v|\lambda')]\\
&= V_{11}(\lambda') - \int_{\mathbf{V}} \bar{q}_1(v,\lambda')[G(v|\lambda) - G(v|\lambda')]dv,
\end{aligned}
$$

where the inequality follows from (IC_{11}) and (SD), and the final equality is by integration by parts. Reversing the roles of λ and λ' and then adding the resulting inequality to the above yields

$$\int_{\mathbf{V}}[\bar{q}_1(v,\lambda) - \bar{q}_1(v,\lambda')][G(v|\lambda) - G(v|\lambda')]dv \leq 0. \tag{B.1}$$

Now, note that for all μ, μ', we can write

$$
\begin{aligned}
\int_{\mathbf{V}} \bar{q}_1(v,\mu)G(v|\mu')dv &= \int_{\mathbf{V}}[x_1(\mu)\bar{q}_{12}(v,\mu) + (1 - x_1(\mu))q_2(v)]G(v|\mu')dv\\
&= x_1(\mu)\int_{\bar{k}_{12}(\mu)}^{\bar{v}} G(v|\mu')dv + (1 - x_1(\mu))\int_{\alpha}^{\bar{v}} G(v|\mu')dv,
\end{aligned}
$$

where $\bar{k}_{12}(\mu)$ and α are the cutoffs corresponding to $\bar{q}_{12}(\cdot,\mu)$ and q_2, respectively. Recall from our previous result, however, that $\bar{q}_{12}(v,\lambda) = q_2(v)$ when $x_1(\lambda) \in (0,1)$. Therefore, we can write

$$\int_{\mathbf{V}} \bar{q}_1(v,\mu)G(v|\mu')dv = \int_{\bar{k}_1(\mu)}^{\bar{v}} G(v|\mu')dv,$$

where we define $\bar{k}_1(\mu) := x_1(\mu)\bar{k}_{12}(\mu) + (1 - x_1(\mu))\alpha$. We can now rewrite (B.1) as

$$\int_{\bar{k}_1(\lambda)}^{\bar{k}_1(\lambda')}[G(v|\lambda) - G(v|\lambda')]dv \leq 0.$$

Since Assumption 1 orders $\{G(\cdot|\mu)\}_{\mu\in\Lambda}$ by first-order stochastic dominance, this inequality holds only if $\bar{k}_1(\cdot)$ is (weakly) decreasing (implying the effective allocation rule $\bar{q}_1$ is nondecreasing). ■

Finally, we turn to the seller's period-two problem ($\widetilde{\text{SR}}$) when cohort-one buyers are free to recontract in period two. This problem is somewhat more subtle than (SR), as the seller's choice of period-two mechanism influences—through constraint ($\widetilde{\text{RC}}$) and its impact on $x_2(v,\lambda)$—the set of buyers that choose to recontract. We show, however, that it is inefficient to induce recontracting using a second-period subsidy: the seller can more cost-effectively induce the *same* ex post sorting by modifying the set of initial-period contracts. Therefore, the optimal period-two contract is simply a price with no additional subsidy. Furthermore, cohort-one buyers will choose to recontract whenever this period-two price is lower than their already-contracted cutoff.

LEMMA B.4. *In an optimal contract, the seller's period-two problem in* ($\widetilde{\text{SR}}$) *can be written as*

$$\max_{\alpha}\left\{\int_{\Lambda}\left(x_1(\lambda)\left[\pi_\lambda(\min\{k_1(\lambda),\alpha\}) - U_{12}(\underline{v},\lambda)\right] + (1 - x_1(\lambda))\pi_\lambda(\alpha)\right)dF(\lambda) + \gamma\pi_H(\alpha)\right\}. \tag{$\widetilde{\text{SR}}'$}$$

PROOF. Using Lemmas B.1 and B.2, the seller's period-two problem ($\widetilde{\text{SR}}$) becomes one of choosing a cutoff α and subsidy $U_{22}(\underline{v})$ to solve

$$\max_{\alpha, U_{22}(\underline{v})} \left\{ \begin{aligned} &\int_\Lambda x_1(\lambda)[\pi_\lambda(\bar{k}_{12}(\lambda)) - V_{12}(\underline{v},\lambda)]dF(\lambda) \\ &\quad + \int_\Lambda (1 - x_1(\lambda))[\pi_\lambda(\alpha) - U_{22}(\underline{v})]dF(\lambda) + \gamma[\pi_H(\alpha) - U_{22}(\underline{v})] \end{aligned} \right\} \tag{B.2}$$

subject to (IR) and ($\widetilde{\text{RC}}$),

where $\bar{k}_{12}$ is the effective cutoff associated with $\bar{q}_{12}(v,\lambda) = x_2(v,\lambda)q_1(v,\lambda) + (1 - x_2(v,\lambda))q_2(v)$. Our first goal is to characterize, given a period-two cutoff $\alpha \in \mathbf{V}$ and subsidy $u := U_{22}(\underline{v}) \geq 0$, the behavior of $\bar{k}_{12}(\lambda)$ and determine which buyers recontract. In doing so, we rely on the observation that $\tilde{p}(\mu) = U_{12}(\underline{v},\mu)$ for all $\mu \in \Lambda$. (This follows from (IR) and the "bang-bang" nature of payments implementing a cutoff allocation.)

Fix an arbitrary $\lambda \in \Lambda$, and observe that $\bar{q}_{12}(v,\lambda) = 1$ for all $v \geq \max\{k_1(\lambda), \alpha\}$, and $\bar{q}_{12}(v,\lambda) = 0$ for all $v < \min\{k_1(\lambda), \alpha\}$. So to determine the behavior of $\bar{q}_{12}(v,\lambda)$ when $\min\{k_1(\lambda), \alpha\} \leq v < \max\{k_1(\lambda), \alpha\}$, suppose first that $k_1(\lambda) \geq \alpha$. Then for any $v \in (\alpha, k_1(\lambda))$, we have

$$\begin{aligned} U_{12}(v,\lambda) - \widetilde{U}_{22}(v) &= (U_{12}(\underline{v},\lambda) + \max\{v - k_1(\lambda), 0\}) - (\tilde{p}(\lambda) + u + \max\{v - \alpha, 0\}) \\ &= 0 - (v - \alpha + u) < 0. \end{aligned}$$

This implies that $x_2(v,\lambda) = 0$ and, therefore, $\bar{q}_{12}(v,\lambda) = 1$ for all $v \in (\alpha, k_1(\lambda))$. (If $v = \alpha$, then the buyer is indifferent between the two contracts and so $\bar{q}_{12}(\alpha,\lambda)$ is indeterminate. Note that this is a zero-probability event, however, and can therefore be safely ignored.)

Now suppose instead that $k_1(\lambda) \in [\alpha - u, \alpha)$. Then for any $v \in [k_1(\lambda), \alpha)$, we have

$$\begin{aligned} U_{12}(v,\lambda) - \widetilde{U}_{22}(v) &= (U_{12}(\underline{v},\lambda) + \max\{v - k_1(\lambda), 0\}) - (U_{12}(\underline{v},\lambda) + u + \max\{v - \alpha, 0\}) \\ &= (v - k_1(\lambda)) - (0 + u) = v - (k_1(\lambda) + u) \leq v - \alpha < 0. \end{aligned}$$

This implies that $x_2(v,\lambda) = 0$ and, therefore, $\bar{q}_{12}(v,\lambda) = 0$ for all $v \in [k_1(\lambda), \alpha)$.

Finally, suppose that $k_1(\lambda) < \alpha - u$. Then for any $v \in [k_1(\lambda), \alpha)$, we have

$$\begin{aligned} U_{12}(v,\lambda) - \widetilde{U}_{22}(v) &= (U_{12}(\underline{v},\lambda) + \max\{v - k_1(\lambda), 0\}) - (U_{12}(\underline{v},\lambda) + u + \max\{v - \alpha, 0\}) \\ &= (v - k_1(\lambda)) - (0 + u) = v - (k_1(\lambda) + u). \end{aligned}$$

This is strictly negative if $v < k_1(\lambda) + u$, in which case $x_2(v,\lambda) = 0$ and $\bar{q}_{12}(v,\lambda) = q_2(v) = 0$; on the other hand, this expression is strictly positive if $v > k_1(\lambda) + u$, in which case $x_2(v,\lambda) = 1$ and $\bar{q}_{12}(v,\lambda) = q_1(v,\lambda) = 1$. (If $v = k_1(\lambda) + U_{22}(\bar{v})$, then the buyer is indifferent and $\bar{q}_{12}(v,\lambda)$ is indeterminate. Of course, this is a zero-probability event and so can be safely ignored.)

Thus, the cutoff associated with $\bar{q}_{12}(v,\lambda)$ is $\bar{k}_{12}(\lambda) := \min\{\alpha, k_1(\lambda) + u\}$, so (B.2) becomes

$$\max_{\alpha, u} \left\{ \begin{aligned} &\int_\Lambda (1 - x_1(\lambda))\pi_\lambda(\alpha)dF(\lambda) + \gamma\pi_H(\alpha) \\ &\quad + \int_\Lambda x_1(\lambda)(\pi_\lambda(\min\{\alpha, k_1(\lambda) + u\}) - U_{12}(\underline{v},\lambda))dF(\lambda) - (1+\gamma)u \end{aligned} \right\} \tag{B.3}$$

subject to $u \geq 0$.

Now fix a candidate optimal contract (denoted by a * superscript), and suppose that the solution (α^*, u^*) to (B.3) is such that $u^* > 0$. Define a new contract (denoted by a ** superscript) by

$x_1^{**}(\lambda) := x_1^*(\lambda)$; $p_{11}^{**}(\lambda) := p_{11}^*(\lambda)$; and $k_1^{**}(\lambda) := k_1^*(\lambda) + u^*$ for all $\lambda \in \Lambda$, and

$$q_1^{**}(v,\lambda) := \begin{cases} 0 & \text{if } v < k_1^{**}(\lambda), \\ 1 & \text{if } v \geq k_1^{**}(\lambda); \end{cases} \text{ and } p_{12}^{**}(v,\lambda) = q_1^{**}(v,\lambda)k_1^{**}(\lambda) + p_{12}^{*} \text{ for all } \lambda \in \Lambda \text{ and } v \in \mathbf{V}.$$

Denoting the objective function in problem (B.3) by $\Pi_2(\alpha, u|x_1, k_1, U_{12})$, we then have

$$\begin{aligned}
&\Pi_2(\alpha^*, 0|x_1^{**}, k_1^{**}, U_{12}^{**}) - \Pi_2(\alpha^* + \epsilon, \delta|x_1^{**}, k_1^{**}, U_{12}^{**}) \\
&\quad = \int_\Lambda (1 - x_1^{**}(\lambda))[\pi_\lambda(\alpha^*) - \pi_\lambda(\alpha^* + \epsilon)]dF(\lambda) + \gamma[\pi_H(\alpha^*) - \pi_H(\alpha^* + \epsilon)] \\
&\qquad + \int_\Lambda x_1^{**}(\lambda)[\pi_\lambda(\min\{\alpha^*, k_1^{**}(\lambda)\}) - \pi_\lambda(\min\{\alpha^* + \epsilon, k_1^{**}(\lambda) + \delta\})]dF(\lambda) + (1+\gamma)\delta \\
&\quad = \int_\Lambda (1 - x_1^{*}(\lambda))[\pi_\lambda(\alpha^*) - \pi_\lambda(\alpha^* + \epsilon)]dF(\lambda) + \gamma[\pi_H(\alpha^*) - \pi_H(\alpha^* + \epsilon)] \\
&\qquad + \int_\Lambda x_1^{*}(\lambda)[\pi_\lambda(\min\{\alpha^*, k_1^{*}(\lambda) + u^*\}) - \pi_\lambda(\min\{\alpha^* + \epsilon, k_1^{*}(\lambda) + u^* + \delta\})]dF(\lambda) + (1+\gamma)\delta \\
&\quad = \Pi_2(\alpha^*, u^*|x_1^{*}, k_1^{*}, U_{12}^{*}) - \Pi_2(\alpha^* + \epsilon, u^* + \delta|x_1^{*}, k_1^{*}, U_{12}^{*}) \geq 0
\end{aligned}$$

for all $\epsilon, \delta \in \mathbb{R}$, where the inequality follows from the (assumed) optimality of (α^*, u^*). Thus, $(\alpha^{**}, u^{**}) = (\alpha^*, 0)$ solves problem (B.3) given the revised (**) period-one contract.

Since this new contract effectively reduces the utility of *all* buyers across *both* cohorts by $u^* > 0$ (and also reduces the value of delay by u^*) while keeping allocations unchanged, the new ** contract satisfies all the period-one constraints and yields greater profits than the original * contract, contradicting the latter's optimality. Thus, we may conclude that in any optimal contract, the period-two contract (on the equilibrium path) corresponds to a price α with no additional subsidies. Thus, in an optimal contract, the seller's problem (B.3) can be written as

$$\max_\alpha \left\{ \int_\Lambda \left(x_1(\lambda) \left[\pi_\lambda(\min\{k_1(\lambda), \alpha\}) - U_{12}(\underline{v}, \lambda) \right] + (1 - x_1(\lambda))\pi_\lambda(\alpha) \right) dF(\lambda) + \gamma \pi_H(\alpha) \right\}. \quad \blacksquare$$

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